TEO (ESIST. E REGOLARITÀ C^1)

$$\mathcal{L}(u) = \int_a^b L(x,u,u')\,dx \qquad A = \{u \in W^{1,p} : u(a)=\alpha,\ u(b)=\beta\}$$

$1 < p < +\infty$

- L CONTINUA E C^1 IN (y,z)
- $|L_y| + |L_z| \leq C(1+|y|^p+|z|^p)$
- $z \to L(x,y,z)$ STRETT. CONVESSA
- $L(x,y,z) \geq c|z|^p + d, \quad c>0$

$\Rightarrow \exists \min \bar{u}$ di $\mathcal{L}$ IN A E $\bar{u} \in C^1$.

OSS: VALE L'EQ. DI E.L. $\dfrac{d}{dx}\left[L_z(x,\bar{u}(x),\bar{u}'(x))\right] = L_y(x,\bar{u}(x),\bar{u}'(x))$

$x \in (a,b)$

TEO (ESISTENZA DI TONELLI) $\qquad \mathcal{L}(u) = \int_a^b L(x,u,u')$

- L CONTINUA
- L DIFF. IN z E L_z CONTINUA
- $z \to L(x,y,z)$ CONVESSA $\forall x,y$
- $L(x,y,z) \geq \psi(z)$ $\quad \psi$ SUPERLINEARE,

CIOÈ $\displaystyle \lim_{z \to \pm\infty} \frac{\psi(z)}{z} = +\infty$

$\Rightarrow \exists$ min di $\mathcal{L}$ IN $\mathcal{A} = \{ u \in W^{1,1} : u(a) = \alpha , u(b) = \beta \}$.

OSS: NON SI APPLICA A $\mathcal{L}(u) = \int \sqrt{1+u'^2} + g(u)$

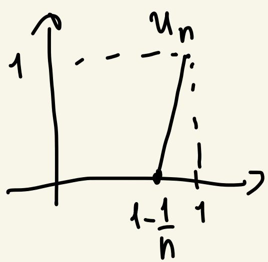
ESEMPI E CONTROESEMPI:

1) $\mathcal{L}(u) = \int_0^1 \sqrt{u^2+u'^2}$ $\qquad (p=1)$

$$A = \{ W^{1,1} : u(0)=0 , \ u(1)=1 \}$$

$$\mathcal{L}(u) > \int_0^1 |u'| \geq \left| \int_0^1 u' \right| = 1 \qquad \forall u \in A$$

VEDIAMO CHE $\inf_A \mathcal{L} = 1 \Rightarrow \nexists$ min di $\mathcal{L}$ IN A



$$u_n(x) = \begin{cases} 0 & x \in [0, 1-\frac{1}{n}] \\ nx-n+1 & x \in [1-\frac{1}{n}, 1] \end{cases}$$

$$\mathcal{L}(u_n) = \int_{1-\frac{1}{n}}^{1} \sqrt{(nx-n+1)^2 + n^2} \leq \int_{1-\frac{1}{n}}^{1} \sqrt{1+n^2} = \sqrt{1+\frac{1}{n^2}} \xrightarrow[n]{} 1$$

2) $\mathcal{L}(u) = \int_0^1 x^2 u'^2 \qquad p = 2$

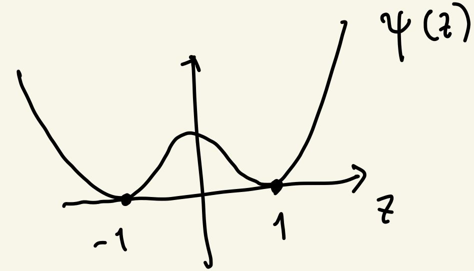
$$A = \{u \in W^{1,2} : u(0) = 1,\ u(1) = 0\}$$

$$u_n(x) = \begin{cases} 1 & x \in \left[0, \frac{1}{n}\right] \\ -\dfrac{\log x}{\log n} & x \in \left[\frac{1}{n}, 1\right] \end{cases}$$

$$0 < \mathcal{L}(u_n) = \int_{\frac{1}{n}}^{1} x^2 \frac{1}{x^2 (\log n)^2} \leq \frac{1}{(\log n)^2} \xrightarrow[n]{} 0 \qquad \text{MA } \mathcal{L}(u) > 0 \ \forall u \in A$$

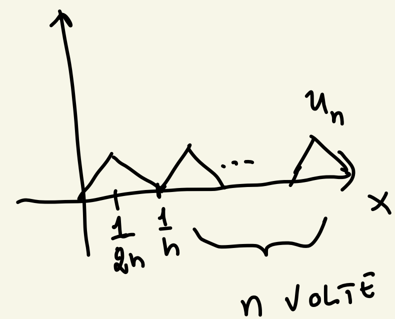
3) (BOLZA)

$$\mathcal{L}(u) = \int_0^1 \underbrace{(1-u'^2)^2}_{\psi(u')} + u^2$$



$$\mathcal{A} = W_0^{1,2}((0,1))$$

$$\mathcal{L}(u) > 0 \quad \forall u \in \mathcal{A}$$



$$u_n'(x) = \pm 1 \quad \tilde{\forall} x$$

$$\|u_n\|_{L^\infty} = \frac{1}{n} \to 0$$

$$\mathcal{L}(u_n) = \int_0^1 u_n^2 \leq \frac{1}{n^2} \xrightarrow[n]{} 0$$

4) $\mathcal{L}(u) = \int_0^1 u'^2 + (u-f)^2$ $\qquad f \in L^2$

$A = \{u \in W^{1,2} : u(0) = \alpha,\ u(1) = \beta\}$

$\exists !$ min di $\mathcal{L}$ in A

SE f È $C^0 \Rightarrow$ IL MINIMO È C^1 E SI HA

$$\frac{d}{dx}[u'] = \underbrace{u-f}_{\in C^0} \Rightarrow u \in C^2 \text{ E VERIFICA } \boxed{u'' = u - f}$$

5) (MANIÀ)

$$\mathcal{L}(u) = \int_0^1 (u^3 - x)^2 u'^6$$

NON È COERCIVO

$u = \sqrt[3]{x} \Rightarrow L(x,u,z) = 0$

$$\mathcal{L}(u) = 0 \iff u = \sqrt[3]{x} \in W^{1,p} \quad \forall p < \frac{3}{2}$$

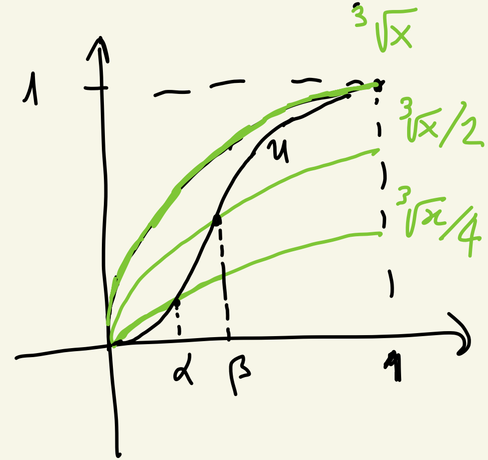
$$u' = \frac{1}{3} x^{-\frac{2}{3}} \qquad \int_0^1 |u'|^p \sim \int_0^1 \frac{1}{x^{2p/3}} < +\infty \iff p < \frac{3}{2}$$

$\exists!$ min DI $\mathcal{L}$ IN $W^{1,p}$ $\forall\, 1 \leq p < \frac{3}{2}$ ED È $u(x) = \sqrt[3]{x} \notin C^1$.

PROP. (FENOMENO DI LAVRENTIEV)

$\exists\, c > 0$ T.C. $\forall u \in Lip$ CON $u(0) = 0$, $u(1) = 1$ SI HA $\mathcal{L}(u) \geq c > 0$.

DIM



$$u \in Lip([0,1])$$

$$u(0)=0 \qquad u(1)=1$$

$$0<\alpha<\beta<1 \quad \text{T.C.}$$

$$u(\alpha)=\sqrt[3]{\alpha}/4, \quad u(\beta)=\sqrt[3]{\beta}/2,$$

$$\boxed{\sqrt[3]{x}/4 \le u(x) \le \sqrt[3]{x}/2 \quad \forall x \in [\alpha,\beta]}$$

$$\mathcal{L}(u) = \int_0^1 (u^3-x)^2 u'^6 \ge \int_\alpha^\beta x^2\left(\frac{u^3}{x}-1\right)^2 u'^6 \underset{\substack{\uparrow \\ \text{OSS}}}{\ge} \left(\frac{7}{8}\right)^2 \int_\alpha^\beta x^2 u'^6 \, dx$$

OSS: $x \in [\alpha,\beta] \Rightarrow \dfrac{1}{64} \le \dfrac{u^3}{x} \le \dfrac{1}{8}$

$$\tilde{u}(y) = u(x), \quad y = x^{\frac{3}{5}}, \quad x = y^{\frac{5}{3}}, \quad dx = \frac{5}{3} y^{\frac{2}{3}} dy$$

$$u'(x) = \frac{3}{5} y^{-\frac{2}{3}} \tilde{u}'(y) \Longrightarrow$$

$$\mathcal{L}(u) \geq \left(\frac{7}{8}\right)^2 \int_{\alpha^{\frac{3}{5}}}^{\beta^{\frac{3}{5}}} \cancel{y^{10/3}} \left(\frac{3}{5}\right)^6 \cancel{y^{-12/3}} \, \tilde{u}'^{\,6} \cdot \frac{5}{3} \cancel{y^{2/3}} \, dy = K \int_{\alpha^{\frac{3}{5}}}^{\beta^{\frac{3}{5}}} |\tilde{u}'|^6 \, dy$$

$$K = \left(\frac{7}{8}\right)^2 \cdot \left(\frac{3}{5}\right)^5$$

PER JENSEN ABBIAMO

$$\mathcal{L}(u) \geq K\left(\beta^{\frac{3}{5}} - \alpha^{\frac{3}{5}}\right) \fint_{\alpha^{\frac{3}{5}}}^{\beta^{\frac{3}{5}}} |\tilde{u}'|^6 \underset{J}{\geq} K\left(\beta^{\frac{3}{5}} - \alpha^{\frac{3}{5}}\right) \left(\fint_{\alpha^{\frac{3}{5}}}^{\beta^{\frac{3}{5}}} \tilde{u}' \right)^6$$

$$= K \frac{\left[u(\beta) - u(\alpha)\right]^6}{\left(\beta^{\frac{3}{5}} - \alpha^{\frac{3}{5}}\right)^5} = K \frac{\left[\frac{\beta^{\frac{1}{3}}}{2} - \frac{\alpha^{\frac{1}{3}}}{4}\right]^6}{\left(\beta^{\frac{3}{5}} - \alpha^{\frac{3}{5}}\right)^5} = \frac{K \cancel{\beta^2} \left(\frac{1}{2} - \frac{1}{4}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{3}}\right)^6}{\cancel{\beta^3} \left(1 - \left(\frac{\alpha}{\beta}\right)^{\frac{3}{5}}\right)^5}$$

$$\geq \frac{K}{2^6} \cdot \frac{\left[1 - \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{3}}\right]^6}{\left(1 - \left(\frac{\alpha}{\beta}\right)^{\frac{3}{5}}\right]^5} \geq \frac{K}{2^6}\Big[\underbrace{1 - \frac{1}{2}\left(\frac{\alpha}{\beta}\right)}_{\geq \frac{1}{2}}^{\frac{1}{3}}\Big]^6 \geq \frac{K}{4^6} > 0$$

$$0 \leq \frac{\alpha}{\beta} \leq 1$$